The Product Sheet

Identities, transformations and convergence tests for $\prod$

0. DEFINITION & CONVENTIONS

$$\prod_{k=m}^{n} a_k = a_m\, a_{m+1} \cdots a_n, \qquad \prod_{k \in S} a_k$$

- **Empty product** $= 1$ whenever $n < m$ or $S = \varnothing$. This is a *convention that makes every other identity work*; never treat it as 0.
- Singleton: $\prod_{k=m}^{m} a_k = a_m$. Splitting is then automatic.
- $\prod$ is the multiplicative sibling of $\sum$: everything you know about sums transfers through log, provided all $a_k > 0$.
- Over a finite index set, order never matters (commutativity). Over matrices/operators it *does* — see the ordered-product box.

1. ALGEBRA OF FINITE PRODUCTS

$$\prod_k (a_k b_k) = \Big(\prod_k a_k\Big)\Big(\prod_k b_k\Big)$$

$$\prod_{k=m}^{n} c\, a_k = c^{\,n-m+1} \prod_{k=m}^{n} a_k \qquad \textit{count the factors!}$$

$$\prod_k a_k^p = \Big(\prod_k a_k\Big)^p, \qquad \prod_{k=m}^{n} c^k = c^{\binom{n+1}{2} - \binom{m}{2}}$$

$$\prod_{k=m}^{n} a_k = \prod_{k=m}^{j} a_k \cdot \prod_{k=j+1}^{n} a_k \qquad \textit{split, } m-1 \le j \le n$$

$$\prod_{k=m}^{n} a_k = \prod_{k=m+t}^{n+t} a_{k-t} \qquad \textit{shift}$$

$$\prod_{k=m}^{n} a_k = \prod_{k=m}^{n} a_{m+n-k} \qquad \textit{reflect}$$

$$\prod_i \prod_j a_{ij} = \prod_j \prod_i a_{ij} \qquad \textit{Fubini}$$

$$\prod_{i=1}^{m} \prod_{j=1}^{n} a_i b_j = \Big(\prod_i a_i\Big)^n \Big(\prod_j b_j\Big)^m$$

Reindex by a bijection. If $\sigma : S \to T$ is a bijection then $\prod_{k \in S} a_{\sigma(k)} = \prod_{t \in T} a_t$. Most "clever" product evaluations are a bijection in disguise (pairing $k \leftrightarrow n-k$, $d \leftrightarrow n/d$, $k \leftrightarrow 2k$ / $2k+1$).

2. LOG–EXP DUALITY: THE MASTER MOVE

For $a_k > 0$,

$$\prod_k a_k = \exp\Big(\sum_k \log a_k\Big), \qquad \log \prod_k a_k = \sum_k \log a_k.$$

Every sum technique becomes a product technique: partial fractions → telescoping factors, Abel summation, Euler–Maclaurin, dominated convergence, Cauchy criterion.

Logarithmic derivative (turns products into sums of poles):

$$\frac{d}{dz} \log \prod_k f_k(z) = \frac{\left(\prod_k f_k\right)'}{\prod_k f_k} = \sum_k \frac{f_k'(z)}{f_k(z)}.$$

Generalised product rule: $\displaystyle\Big(\prod_{k=1}^{n} f_k\Big)' = \sum_{j=1}^{n} f_j' \prod_{k \ne j} f_k.$

Geometric mean / AM–GM:

$$\Big(\prod_{k=1}^{n} a_k\Big)^{1/n} \le \frac{1}{n} \sum_{k=1}^{n} a_k, \qquad a_k \ge 0,$$

with equality iff all a_k are equal. Weighted form: $\prod a_k^{w_k} \le \sum w_k a_k$ for $\sum w_k = 1$, $w_k \ge 0$ (this *is* Young/Hölder).

Product integral (continuous limit): $\displaystyle\prod_a^b (1 + f(x)\,dx) = \exp \int_a^b f(x)\,dx.$

3. TELESCOPING — THE SINGLE MOST USEFUL TRICK

If $a_k = \dfrac{f(k+1)}{f(k)}$ then

$$\boxed{\prod_{k=m}^{n} \frac{f(k+1)}{f(k)} = \frac{f(n+1)}{f(m)}}$$

Strategy: factor a_k and look for a shift $f(k) \mapsto f(k+1)$ or $f(k+j)$. A ratio of polynomials telescopes iff numerator and denominator roots differ by integers.

Standard telescopes:

$$\prod_{k=1}^{n}\Big(1 + \frac{1}{k}\Big) = n+1, \qquad \prod_{k=2}^{n}\Big(1 - \frac{1}{k}\Big) = \frac{1}{n}$$

$$\prod_{k=2}^{n}\Big(1 - \frac{1}{k^2}\Big) = \frac{n+1}{2n}, \qquad \prod_{k=2}^{\infty}\Big(1 - \frac{1}{k^2}\Big) = \frac{1}{2}$$

$$\prod_{k=0}^{n-1}\big(1 + x^{2^k}\big) = \frac{1 - x^{2^n}}{1 - x} \qquad \textit{binary}$$

$$\prod_{k=1}^{n} \cos\frac{x}{2^k} = \frac{\sin x}{2^n \sin(x/2^n)} \xrightarrow[n\to\infty]{} \frac{\sin x}{x}$$

$$\prod_{k=2}^{n} \frac{k^3 - 1}{k^3 + 1} = \frac{2(n^2 + n + 1)}{3n(n+1)} \xrightarrow[n\to\infty]{} \frac{2}{3}$$

The last one: $k^3-1 = (k-1)(k^2+k+1)$, $k^3+1 = (k+1)(k^2-k+1)$, and $k^2-k+1 = f(k-1)$ for $f(k) = k^2+k+1$ — two telescopes at once.

Multiplicative difference calculus. With $(\Delta_\times f)(k) = f(k+1)/f(k)$, the fundamental theorem reads $\prod_{k=m}^{n} (\Delta_\times f)(k) = f(n+1)/f(m)$, and "multiplicative summation by parts" is

$$\prod_k f(k)^{\Delta g(k)} = \big(\text{boundary}\big) \Big/ \prod_k g(k+1)^{\Delta \log f(k)}.$$

4. FACTORIALS, FALLING & RISING POWERS

$$n! = \prod_{k=1}^{n} k, \qquad 0! = 1 \ \textit{(empty product)}$$

$$x^{\underline{n}} = \prod_{k=0}^{n-1}(x-k) = \frac{\Gamma(x+1)}{\Gamma(x-n+1)} \quad \textit{falling}$$

$$(x)_n = x^{\overline{n}} = \prod_{k=0}^{n-1}(x+k) = \frac{\Gamma(x+n)}{\Gamma(x)} \quad \textit{rising}$$

$$\binom{n}{k} = \prod_{j=1}^{k} \frac{n-j+1}{j} = \frac{n^{\underline{k}}}{k!}$$

$$\prod_{k=1}^{n}(2k) = 2^n n! = (2n)!!$$

$$\prod_{k=1}^{n}(2k-1) = \frac{(2n)!}{2^n n!} = (2n-1)!!$$

$$\prod_{k=1}^{n} k! = G(n+2) \quad \textit{Barnes G}$$

$$\prod_{k=1}^{n} k^k = H(n) \quad \textit{hyperfactorial}$$

Vandermonde:

$$\det\left(x_i^{j-1}\right)_{i,j=1}^{n} = \prod_{1 \le i < j \le n} (x_j - x_i).$$

Composition & reflection of falling powers:

$$x^{\underline{m+n}} = x^{\underline{m}} \cdot (x-m)^{\underline{n}}, \qquad x^{\underline{n}} = (-1)^n (-x)^{\overline{n}}.$$

5. GAMMA: THE CONTINUOUS FACTORIAL

$$\Gamma(z+1) = z\,\Gamma(z), \qquad \Gamma(n+1) = n!.$$

Euler's limit / product forms

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\, n^z}{\prod_{k=0}^{n}(z+k)} = \frac{1}{z}\prod_{n=1}^{\infty} \frac{(1+1/n)^z}{1+z/n}$$

$$\frac{1}{\Gamma(z)} = z\, e^{\gamma z} \prod_{n=1}^{\infty}\left(1+\frac{z}{n}\right)e^{-z/n} \quad \textit{Weierstrass}$$

Reflection & duplication

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}$$

$$\Gamma(z)\Gamma\left(z+\tfrac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z) \quad \textit{Legendre}$$

Gauss multiplication — the general product of shifts:

$$\prod_{k=0}^{m-1} \Gamma\left(z+\frac{k}{m}\right) = (2\pi)^{\frac{m-1}{2}}\, m^{\frac{1}{2}-mz}\, \Gamma(mz).$$

Use: any product $\prod_k (ak+b)$ over a range collapses to a ratio of Γ's — $\prod_{k=0}^{n-1}(ak+b) = a^n\,\Gamma(n+b/a)/\Gamma(b/a)$. This is the universal "closed form" for products of linear factors.

6. ROOTS OF UNITY & POLYNOMIAL PRODUCTS

With $\omega = e^{2\pi i/n}$:

$$x^n - 1 = \prod_{k=0}^{n-1}(x-\omega^k), \quad x^n - a^n = \prod_{k=0}^{n-1}(x - a\omega^k)$$

$$\prod_{k=1}^{n-1}(1-\omega^k) = n, \qquad \prod_{k=1}^{n-1} 2\sin\frac{k\pi}{n} = n$$

$$\prod_{k=1}^{n-1}\sin\frac{k\pi}{n} = \frac{n}{2^{n-1}}, \qquad \prod_{k=1}^{n-1}\cos\frac{k\pi}{n} = \frac{\sin(n\pi/2)}{2^{n-1}}$$

Cyclotomic factorisation & Möbius inversion:

$$x^n - 1 = \prod_{d\mid n}\Phi_d(x), \qquad \Phi_n(x) = \prod_{d\mid n}(x^d-1)^{\mu(n/d)}.$$

Vieta: monic $p(x) = \prod_i (x-r_i)$ of degree n has $\prod_i r_i = (-1)^n p(0)$; more generally $p(c) = \prod_i (c-r_i)$ — *evaluate the polynomial* rather than multiply the factors.

Resultant: $\operatorname{Res}(f,g) = a^{\deg g} b^{\deg f} \prod_{i,j}(\alpha_i - \beta_j) = a^{\deg g}\prod_i g(\alpha_i)$.

Determinant: $\det A = \prod_i \lambda_i$, $\det(AB) = \det A \det B$, $\det e^A = e^{\operatorname{tr} A}$.

7. PRODUCTS IN NUMBER THEORY

For $n = \prod_p p^{\nu_p(n)}$ (finitely many $\nu_p \neq 0$):

$$\tau(n) = \prod_{p\mid n}(\nu_p+1), \qquad \sigma_s(n) = \prod_{p\mid n}\frac{p^{s(\nu_p+1)}-1}{p^s-1}$$

$$\varphi(n) = n\prod_{p\mid n}\left(1-\frac{1}{p}\right), \qquad \prod_{d\mid n} d = n^{\tau(n)/2}$$

- f multiplicative $\Rightarrow f(n) = \prod_{p^a \| n} f(p^a)$; the whole theory is "define on prime powers, take products".
- **Wilson:** $\prod_{k=1}^{p-1} k \equiv -1 \pmod{p}$ for p prime.
- **Euler product:** for multiplicative f with $\sum |f(n)| n^{-\sigma} < \infty$,

$$\sum_{n\ge 1}\frac{f(n)}{n^s} = \prod_p\left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \cdots\right).$$

In particular $\zeta(s) = \prod_p (1-p^{-s})^{-1}$, $\Re s > 1$, and $\frac{1}{\zeta(s)} = \prod_p (1-p^{-s}) = \sum_n \frac{\mu(n)}{n^s}$.

- **Mertens III:** $\prod_{p\le x}\left(1-\frac{1}{p}\right)^{-1} \sim e^{\gamma}\log x$.

8. q-PRODUCTS, PARTITIONS, THETA

q-Pochhammer ($|q| < 1$ for the infinite case):

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k), \qquad (a;q)_\infty = \prod_{k\ge 0}(1-aq^k).$$

$$[n]_q! = \prod_{k=1}^{n}\frac{1-q^k}{1-q}, \qquad \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$$

Euler's generating function for partitions

$$\prod_{k\ge 1}\frac{1}{1-q^k} = \sum_{n\ge 0} p(n) q^n, \qquad \prod_{k\ge 1}(1+q^k) = \prod_{k\ge 1}\frac{1}{1-q^{2k-1}}.$$

Pentagonal number theorem

$$\prod_{k\ge1}(1-q^k)=\sum_{j\in\mathbb{Z}}(-1)^j q^{j(3j-1)/2}.$$

Jacobi triple product — the identity behind most of the rest:

$$\prod_{m\ge1}(1-q^{2m})(1+q^{2m-1}z)(1+q^{2m-1}z^{-1})=\sum_{n\in\mathbb{Z}}q^{n^2}z^n.$$

q-binomial theorem: $\displaystyle\sum_{n\ge0}\frac{(a;q)_n}{(q;q)_n}z^n=\frac{(az;q)_\infty}{(z;q)_\infty}$.

Dedekind eta: $\eta(\tau)=q^{1/24}\prod_{n\ge1}(1-q^n)$, $q=e^{2\pi i\tau}$.

9. INFINITE PRODUCTS: CONVERGENCE

Write the product as $\prod_{k\ge1}(1+a_k)$ (always do this — factor out the limiting value first).

Definition. $\prod_{k\ge1}(1+a_k)$ *converges* if the partial products tend to a limit that is **nonzero**, after discarding finitely many factors. Tending to 0 counts as **divergence to zero**.

Necessary. $a_k\to0$. (Not sufficient: $\prod(1+1/k)$ diverges.)

Main criterion. For $a_k\ne-1$, the product converges iff $\sum_k\log(1+a_k)$ converges (any fixed branch, valid once $|a_k|<1$).

Absolute convergence. $\sum_k|a_k|<\infty\iff\prod_k(1+|a_k|)<\infty\Rightarrow\prod(1+a_k)$ converges, and the value is invariant under rearrangement.

Positive terms. For $a_k\ge0$: $\prod(1+a_k)$ converges $\iff\sum a_k<\infty$. For $0\le a_k<1$: $\prod(1-a_k)>0\iff\sum a_k<\infty$; if $\sum a_k=\infty$ then $\prod(1-a_k)=0$.

Second-order test. If $\sum a_k$ converges but $\sum a_k^2$ diverges, the product diverges to 0: $\log(1+a_k)=a_k-\frac12a_k^2+O(a_k^3)$. *Always check the quadratic term.*

Uniform / analytic. If f_k are holomorphic on Ω and $\sum_k\sup_K|a_k|<\infty$ on every compact K, then $\prod(1+a_k)$ converges uniformly on compacta to a holomorphic F; $F(z)=0$ exactly where some factor vanishes, with multiplicity the sum of multiplicities. *(Weierstrass)*

Useful bounds

$$\prod_k(1+a_k)\le\exp\Big(\sum_k a_k\Big)\qquad(a_k\ge0)$$
$$\prod_k(1+a_k)\ge1+\sum_k a_k\qquad(a_k\ge0)$$
$$\prod_k(1-a_k)\ge1-\sum_k a_k\qquad(0\le a_k\le1)$$
$$\Big|\prod_k(1+a_k)-1\Big|\le\exp\Big(\sum_k|a_k|\Big)-1$$

The last one is the workhorse for estimating tails: truncation error of an infinite product is controlled by the tail of $\sum|a_k|$.

10. CLASSICAL INFINITE PRODUCTS

$$\sin\pi z=\pi z\prod_{n=1}^{\infty}\Big(1-\frac{z^2}{n^2}\Big)$$
$$\cos\pi z=\prod_{n=1}^{\infty}\Big(1-\frac{4z^2}{(2n-1)^2}\Big)$$
$$\sinh\pi z=\pi z\prod_{n=1}^{\infty}\Big(1+\frac{z^2}{n^2}\Big)$$
$$\frac{\sin x}{x}=\prod_{k=1}^{\infty}\cos\frac{x}{2^k}$$
$$\frac{\pi}{2}=\prod_{n=1}^{\infty}\frac{4n^2}{4n^2-1}=\frac21\cdot\frac23\cdot\frac43\cdot\frac45\cdots$$
$$\frac{2}{\pi}=\frac{\sqrt2}{2}\cdot\frac{\sqrt{2+\sqrt2}}{2}\cdot\frac{\sqrt{2+\sqrt{2+\sqrt2}}}{2}\cdots$$

$$\prod_{n=2}^{\infty}\Big(1-\frac{1}{n^2}\Big)=\frac12,\qquad\prod_{n=1}^{\infty}\Big(1+\frac{1}{n^2}\Big)=$$
$$\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}=\frac23,\qquad\prod_{n=1}^{\infty}\Big(1-\frac{1}{4n^2}\Big)=$$

Everything in the sin/cos family is one identity: an entire function of order 1 is determined by its zeros up to e^{az+b} (Hadamard). Substituting $z=1/2$, $z=i$, $z=1/4$ into the sine product recovers Wallis, $\sinh\pi/\pi$, and friends.

11. BUILDING FUNCTIONS FROM THEIR ZEROS

Weierstrass elementary factors

$$E_0(w)=1-w,\qquad E_p(w)=(1-w)\exp\Big(\sum_{j=1}^{p}\frac{w^j}{j}\Big),\ p\ge1.$$

The exponential is a convergence factor: $|1-E_p(w)|\le|w|^{p+1}$ for $|w|\le1$.

Weierstrass factorisation. Every entire $f\not\equiv0$ with zeros $a_1,a_2,\dots$ (repeated by multiplicity, $a_n\to\infty$) and a zero of order m at 0 factors as

$$f(z)=z^m e^{g(z)}\prod_{n\ge1}E_{p_n}\Big(\frac{z}{a_n}\Big),$$

g entire, with any p_n making $\sum_n(r/|a_n|)^{p_n+1}<\infty$ for all r.

Hadamard. If f has finite order ρ, one may take $p=\lfloor\rho\rfloor$ uniformly and g a polynomial of degree $\le\rho$:

$$f(z)=z^m e^{g(z)}\prod_{n\ge1}E_p\Big(\frac{z}{a_n}\Big),\qquad\deg g\le\rho.$$

Consequences: convergence exponent $\sum|a_n|^{-s}<\infty$ for $s>\rho$; a nonvanishing entire function of finite order is e^{poly}; the sine product and $1/\Gamma$ are the two canonical examples ($\rho=1$).

12. ORDERED (NONCOMMUTATIVE) PRODUCTS

For matrices / operators, fix a convention and *state it*:

$$\overrightarrow{\prod_{k=1}^{n}}A_k=A_1A_2\cdots A_n,\qquad\overleftarrow{\prod_{k=1}^{n}}A_k=A_n\cdots A_1.$$

- $\det\prod_k A_k=\prod_k\det A_k$ and $(\prod_k A_k)^{-1}=\overleftarrow{\prod_k}A_k^{-1}$ (order reverses).
- tr is *cyclic*, not multiplicative: $\operatorname{tr}(AB)=\operatorname{tr}(BA)$, but

$\operatorname{tr}(AB) \neq \operatorname{tr} A \operatorname{tr} B$.

- $e^A e^B = e^{A+B}$ only if $[A,B]=0$; in general use BCH, or the **Lie–Trotter** product $e^{A+B} = \lim_{n\to\infty} (e^{A/n}e^{B/n})^n$.
- **Product integral / time-ordered exponential.** The solution of $Y'(t) = A(t)Y(t)$, $Y(0) = I$ is $Y(t) = \mathcal{T}\exp\int_0^t A(s)\,ds = \lim_{\Delta\to 0} \overleftarrow{\prod_j} (I + A(t_j)\Delta)$.
- Convergence of $\prod(I + A_k)$: guaranteed if $\sum_k \|A_k\| < \infty$.

13. ASYMPTOTICS OF PRODUCTS

Take logs, then use whatever you know about sums.

$$\log \prod_{k=1}^n f(k) = \sum_{k=1}^n \log f(k)$$
$$\approx \int_1^n \log f(x)\,dx + \tfrac{1}{2}\log f(n) + \cdots$$

(Euler–Maclaurin; the correction terms give the $\sqrt{2\pi n}$ in Stirling.)

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} + \cdots\right)$$
$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}, \qquad \prod_{k=1}^n \left(1 + \frac{x}{k}\right) \sim \frac{n^x}{\Gamma(x+1)}$$

Small-factor linearisation. If $\max_k |a_k| \to 0$ and $\sum_k a_k \to S$ with $\sum_k a_k^2 \to 0$, then $\prod_k (1 + a_k) \to e^S$. *This is the Poisson-approximation engine:* $\prod(1-p_i) \approx e^{-\sum p_i}$.

Sensitivity. $\frac{\delta P}{P} = \sum_k \frac{\delta a_k}{a_k}$ — relative errors *add*. A product of n factors each known to ε is known to $\approx n\varepsilon$.

14. PROBABILITY & GENERATING FUNCTIONS

- Independence: $\mathbb{P}(\bigcap_k E_k) = \prod_k \mathbb{P}(E_k)$; $\mathbb{E}\prod_k X_k = \prod_k \mathbb{E}X_k$ for independent X_k.
- Survival / at least one: $\mathbb{P}(\text{none}) = \prod_k (1 - p_k) \approx e^{-\sum p_k}$ when all p_k small.
- Birthday-type: $\prod_{k=0}^{n-1}\left(1 - \frac{k}{N}\right) \approx e^{-n^2/2N}$.
- MGF/PGF of a sum of independents is the *product* of MGF/PGFs; characteristic function of a convolution likewise.
- Subsets and multisets: $\prod_i (1 + x^{a_i})$ counts subsets by weight; $\prod_i \frac{1}{1 - x^{a_i}}$ counts multisets.
- Second-moment / Lovász-style bounds and Chernoff all reduce to $\prod \mathbb{E}e^{tX_k}$.

15. TRAPS & SANITY CHECKS

- **Empty product is** 1, not 0. Check your boundary case with $n = m - 1$.
- **Count the factors.** $\prod_{k=m}^n c = c^{n-m+1}$ — off-by-one here is the most common error on the sheet.
- **One zero factor kills everything.** Before dividing by a product, verify no factor vanishes (e.g. $\prod_{k=1}^{n-1} \cos(k\pi/n)$ is 0 for even n).
- **Convergence to** 0 **is divergence** for infinite products. $\prod(1 - 1/k) = 0$ "converges" in the naive sense but is a divergent product.
- log **needs care with signs and branches.** For complex or sign-changing factors, split off signs first, or use $\operatorname{Log}(1 + a_k)$ only once $|a_k| < 1$.
- $\sum a_k$ **convergent does not imply** $\prod(1 + a_k)$ **convergent** — check $\sum a_k^2$. Counterexample: $a_k = (-1)^k/\sqrt{k}$.
- **Rearrangement** of a conditionally (non-absolutely) convergent product can change its value, exactly as for series.
- **Ordered products don't commute**; also $\prod_k A_k^{-1} \neq (\prod_k A_k)^{-1}$ unless you reverse.
- **Numerical products underflow.** Accumulate $\sum \log a_k$ instead, or rescale; and prefer $\operatorname{log1p}(a_k)$ when a_k is small.
- **Sanity test any claimed identity at small** n ($n = 1, 2, 3$) and, for infinite products, numerically at 10^3 terms with the tail bound $\exp(\sum |a_k|) - 1$.

16. DICTIONARY: SUMS ↔ PRODUCTS

Sum	Product
$\sum a_k$	$\prod a_k$
0 (empty)	1 (empty)
$\Delta f(k) = f(k{+}1) - f(k)$	$f(k{+}1)/f(k)$
telescoping sum	telescoping product
arithmetic mean	geometric mean
$\sum$ converges	$\prod(1 + a_k)$ converges
absolute convergence $\sum \lvert a_k\rvert$	$\prod(1 + \lvert a_k\rvert) < \infty$
partial fractions	factor & shift
Abel summation	multiplicative parts
$e^{\sum}$	$\prod e^{(\cdot)}$
Dirichlet series $\sum f(n)n^{-s}$	Euler product $\prod_p(\cdots)$
$\log\Gamma$, digamma ψ	Γ, $1/\Gamma$
poles of f'/f	zeros of f

17. RECIPE: ATTACKING AN UNFAMILIAR PRODUCT

1. **Normalise.** Write every factor as $1 + a_k$ with $a_k \to 0$, pulling out constants and the dominant growth first.
2. **Factor completely.** Difference of squares/cubes, $k^2 \pm k + 1$, cyclotomics, sin/cos product formulas. Look for roots differing by an integer.
3. **Hunt a telescope.** Is $a_k = f(k{+}1)/f(k)$ or $f(k{+}j)/f(k)$? Ratios of polynomials with integer-shifted roots always collapse to a Γ-ratio.
4. **Otherwise take logs.** $\sum \log(1 + a_k)$; expand to second order and decide convergence; Euler–Maclaurin for asymptotics.
5. **Recognise the shape.** $\prod(1 - z^2/n^2) \Rightarrow \sin$; $\prod(1 + z/n)e^{-z/n} \Rightarrow 1/\Gamma$; $\prod(1 - q^k) \Rightarrow$ partitions/eta; $\prod_p(\cdots) \Rightarrow$ Dirichlet series.
6. **Verify.** Small n by hand, then numerically with the tail bound $\exp(\sum_{k>N} |a_k|) - 1$.