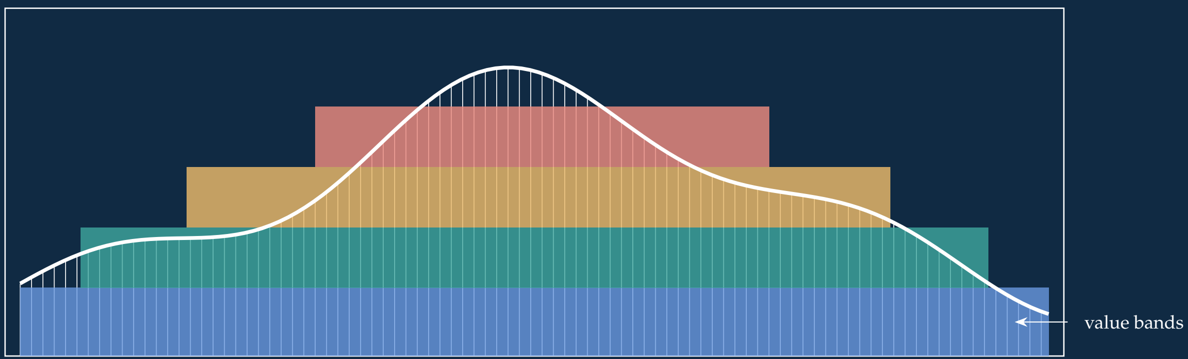
Lebesgue integration

Measure the domain by the values a function takes. Build the integral from measurable sets, then pass safely to limits.



The governing idea

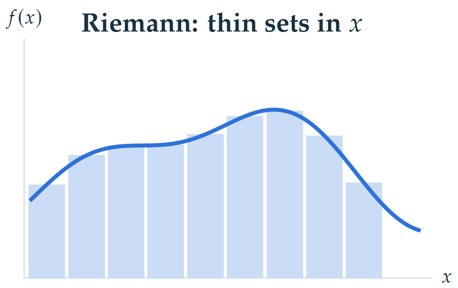
Riemann cuts the *horizontal axis* into intervals. Lebesgue groups points of the domain by *height*. The integral becomes a weighted sum: height × the measure of the set carrying that height.

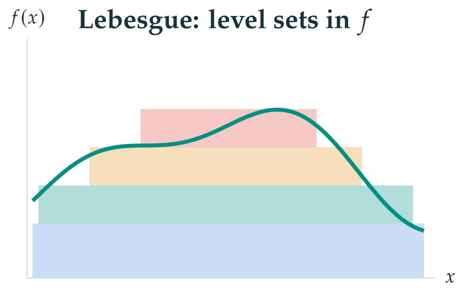
Prerequisites: limits, series, and basic real analysis

Benjamin Michael Haire / 2026

Partition the range, not the domain

Both theories seek area. Their bookkeeping differs: Riemann samples thin vertical strips; Lebesgue measures the sets on which the function has comparable values.





What Riemann needs
Bounded functions with discontinuities confined to a set of measure zero are Riemann integrable. Dense or wild discontinuities can defeat its interval sums.

What Lebesgue gains
It integrates many limits of functions, treats null sets as negligible, and turns convergence into a systematic theory. Every Riemann-integrable function is Lebesgue integrable with the same value.

Layer-cake intuition: $$\int_X f\,\mathrm{d}\mu = \int_0^\infty \mu\{x : f(x) > t\}\,\mathrm{d}t \quad (f \geq 0).$$

Which sets may receive a size?

A measure is not assigned to every imaginable subset. We first select a stable collection of measurable sets, then require size to add over disjoint countable unions.

SPACE
X

→

SIGMA-ALGEBRA
$\Sigma \subseteq \mathcal{P}(X)$

→

MEASURE
$\mu : \Sigma \to [0, \infty]$

σ-algebra axioms
$X \in \Sigma$; if $A \in \Sigma$, then $A^c \in \Sigma$; and if $A_n \in \Sigma$, then $\bigcup_{n=1}^{\infty} A_n \in \Sigma$. Consequently it is also closed under countable intersections and differences.

Measure axioms
$\mu(\varnothing) = 0$ and, for pairwise disjoint A_n,

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n).$$

Monotonicity and countable subadditivity follow.

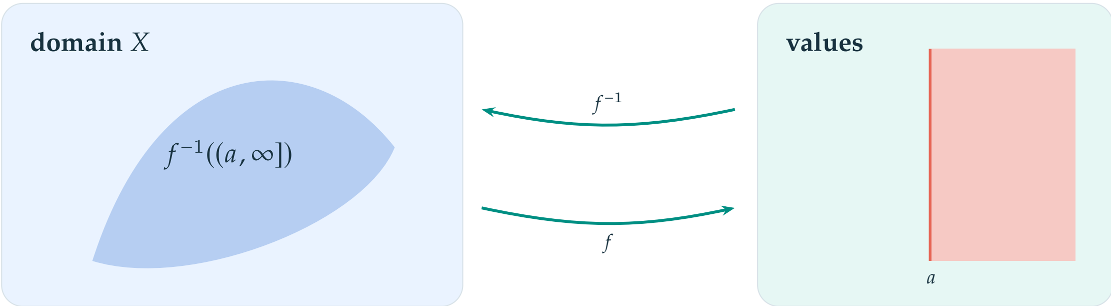
ON THE REAL LINE
The Borel sets $\mathcal{B}(\mathbb{R})$ are generated by open intervals. Lebesgue measure m extends interval length, $m([a, b]) = b - a$, to the completed Borel σ-algebra, including all subsets of null sets. Countable sets have measure zero, yet $\mathbb{R} \setminus \mathbb{Q}$ has full measure in every interval.

Almost everywhere (a.e.). A statement holds almost everywhere when its exceptional set has measure zero. Lebesgue integration identifies functions that differ only on a null set: changing finitely or countably many values does not change the integral.

Measurability is a preimage condition

A function is measurable when asking whether its value falls below a threshold always produces a measurable subset of the domain.

$$f : (X, \Sigma) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \text{ is measurable} \iff \{x : f(x) > a\} \in \Sigma \text{ for every } a \in \mathbb{R}.$$



THE FIRST INTEGRABLE BUILDING BLOCKS

A nonnegative simple function has finitely many heights:

$$s(x) = \sum_{k=1}^{n} a_k \mathbf{1}_{A_k}(x), \qquad a_k \geq 0,\ A_k \in \Sigma.$$

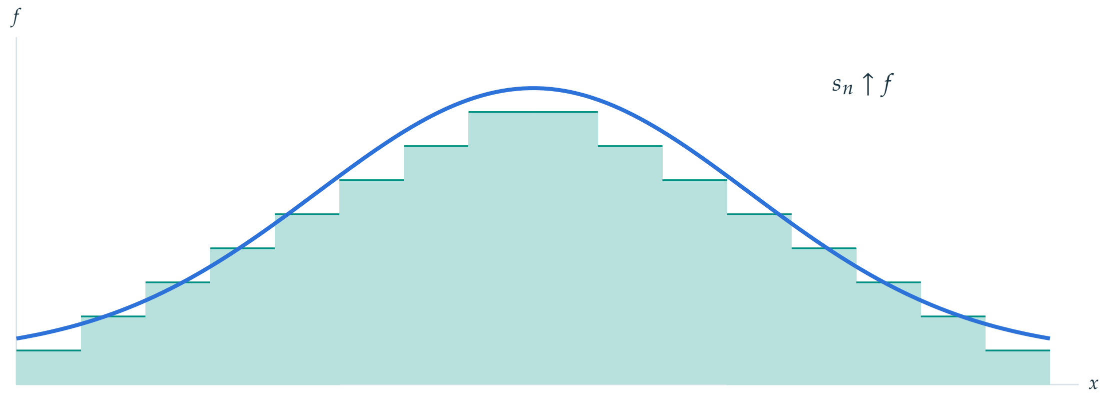
If the A_k are disjoint, define $\int_X s \, \mathrm{d}\mu = \sum_{k=1}^{n} a_k \mu(A_k)$. Different representations give the same value.

Closure facts worth using. Continuous functions are Borel measurable. Sums, products, pointwise suprema, pointwise limits, and positive/negative parts of measurable functions are measurable whenever defined.

Simple functions climb from below

For $f \geq 0$, the integral is the largest area achieved by any measurable step function that never rises above f. This makes approximation part of the definition.

$$\int_X f \, d\mu = \sup\left\{\int_X s \, d\mu : 0 \leq s \leq f,\ s \text{ simple}\right\} \in [0, \infty].$$



Canonical approximation
For $f \geq 0$, quantize heights in steps of 2^{-n} and truncate at n:

$$s_n = \begin{cases} 2^{-n}\lfloor 2^n f \rfloor, & f < n, \\ n, & f \geq n. \end{cases}$$

Then $0 \leq s_n \uparrow f$ pointwise.

Signed functions
Split $f = f^+ - f^-$, where $f^+ = \max(f, 0)$ and $f^- = \max(-f, 0)$. Define

$$\int f = \int f^+ - \int f^-$$

when at least one term is finite. The function is integrable when $\int |f| < \infty$.

Choose the shortest legitimate route

In ordinary calculus problems, Lebesgue integrals often evaluate by familiar antiderivatives. The new theory matters most in deciding existence, ignoring null sets, interchanging limits, and handling irregular functions.

1. Specify (X, Σ, μ) and the domain → **2. Check** measurability and sign → **3. Select** a route below → **4. Justify** finiteness or limit exchange

Route A: simple-function sum
If $f = \sum a_k \mathbf{1}_{A_k}$ on disjoint measurable sets, use $\sum a_k \mu(A_k)$.

Route B: ordinary calculus
For a piecewise continuous f on an interval, compute the usual improper/Riemann integral, provided $\int |f| < \infty$.

Route C: distribution function
For $f \geq 0$, use $\int_0^\infty \mu(f > t)\,\mathrm{d}t$. Excellent for indicators, powers, and tails.

Route D: approximation
Choose $s_n \uparrow f$ and invoke monotone convergence.

Route E: limit under the integral
Seek domination by an integrable g, then use dominated convergence.

Route F: split sign or region
Use f^+, f^-, symmetry, disjoint supports, or a change of variables. Never write $\infty - \infty$.

Proof boundary. Antiderivatives calculate a candidate value. Measurability and absolute integrability justify that the Lebesgue integral exists as a finite number. A convergence theorem justifies moving a limit through $\int$.

Two foundational examples

Indicators turn geometry into algebra. They also expose immediately why dense discontinuity is not the same thing as large measure.

EXAMPLE 1: A SIMPLE FUNCTION

On $[0, 4]$ with Lebesgue measure, let

$$s = 3\mathbf{1}_{[0,1)} + \tfrac{1}{2}\mathbf{1}_{[1,3]} + 2\mathbf{1}_{(3,4]}.$$

The sets are disjoint, so

$$\begin{aligned}\int_0^4 s\,\mathrm{d}x &= 3\,m[0,1) + \tfrac{1}{2}m[1,3] + 2m(3,4] \\ &= 3(1) + \tfrac{1}{2}(2) + 2(1) = \boxed{6}.\end{aligned}$$

Endpoints are irrelevant because finite sets are null.

EXAMPLE 2: DIRICHLET'S FUNCTION

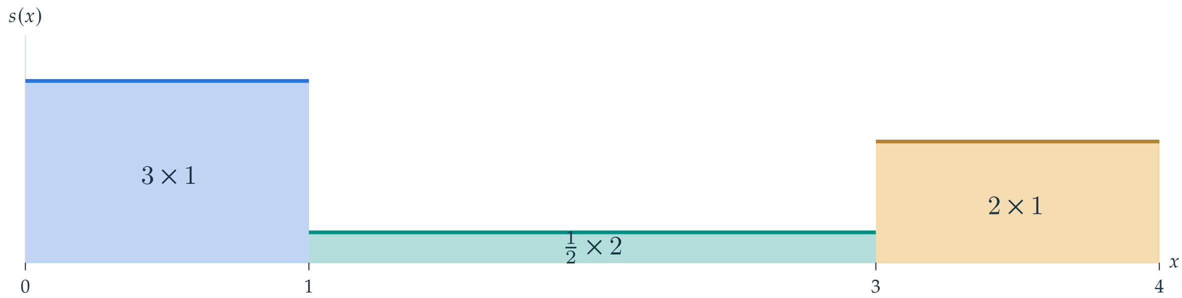
Let $f = \mathbf{1}_{\mathbb{Q}\cap[0,1]}$. It is discontinuous at every point, so it is not Riemann integrable. But $\mathbb{Q} \cap [0, 1]$ is countable:

$$m(\mathbb{Q} \cap [0,1]) = 0.$$

Therefore

$$\int_0^1 f\,\mathrm{d}x = 1 \cdot 0 = \boxed{0}.$$

Meanwhile $f = 0$ almost everywhere. The integral sees measure, not topological density.



What happens near infinity?

Lebesgue theory handles unbounded functions naturally: truncate, integrate the bounded pieces, and pass upward by monotone convergence.

EXAMPLE 3: A POWER SINGULARITY

For $f(x) = x^{-p}$ on $(0, 1)$, $p > 0$, define $f_n = \min(f, n)$. Then $f_n \uparrow f$, so

$$\int_0^1 x^{-p}\,\mathrm{d}x = \lim_{\varepsilon \downarrow 0} \int_\varepsilon^1 x^{-p}\,\mathrm{d}x.$$

If $p \neq 1$, this equals

$$\lim_{\varepsilon \downarrow 0} \frac{1-\varepsilon^{1-p}}{1-p}.$$

Hence

$$\boxed{\int_0^1 x^{-p}\,\mathrm{d}x < \infty \iff p < 1}$$

and, when $p < 1$, $$\int_0^1 x^{-p}\,\mathrm{d}x = \frac{1}{1-p}.$$

For $p = 1$, the logarithm diverges.

EXAMPLE 4: LAYER CAKE

Let $f(x) = e^{-x}$ on $[0, \infty)$. For $0 < t < 1$,

$$\{x : e^{-x} > t\} = [0, -\log t),$$

so its measure is $-\log t$. For $t \geq 1$ it is empty. Thus

$$\int_0^\infty e^{-x}\,\mathrm{d}x = \int_0^1 (-\log t)\,\mathrm{d}t$$
$$= \boxed{1}.$$

The same area is read from the widths of horizontal superlevel sets.

Tail test. On $(1, \infty)$, $\int_1^\infty x^{-p}\,\mathrm{d}x < \infty$ exactly when $p > 1$. Near zero the threshold is $p < 1$; at infinity it reverses. Always identify where the possible divergence lives.

When may a limit pass through the integral?

Pointwise convergence alone is too weak. These theorems provide three progressively different forms of control.

Theorem	Hypotheses	Conclusion
Monotone convergence	$0 \le f_n \uparrow f$ a.e.	$\int f_n \to \int f$; infinity is allowed.
Fatou's lemma	$f_n \ge 0$ measurable	$\int \liminf f_n \le \liminf \int f_n$.
Dominated convergence	$f_n \to f$ a.e. and $\lvert f_n \rvert \le g$ with $g \in L^1$	$f \in L^1$, $\int f_n \to \int f$, and $\lVert f_n - f \rVert_1 \to 0$.

Increasing?
Use MCT

→

Uniform integrable bound?
Use DCT

→

Only nonnegativity?
Fatou gives an inequality

WHY POINTWISE CONVERGENCE FAILS

On $(0,1)$ let $f_n = n\mathbf{1}_{(0,1/n)}$. Then $f_n(x) \to 0$ for every $x > 0$, but $\int_0^1 f_n \, dx = n(1/n) = 1$. The mass grows taller while concentrating near zero. No single integrable function dominates the entire sequence.

Memory cue. MCT controls direction; DCT controls size; Fatou gives the one-sided statement that survives with minimal control.

Two complete limit exchanges

The essential move is not evaluating the integral; it is naming the theorem and verifying every hypothesis.

EXAMPLE 5: DOMINATED CONVERGENCE

Compute $\lim_{n\to\infty}\int_0^1 \frac{x^n}{1+x}\,dx$. For $0 \le x < 1$, $x^n/(1+x) \to 0$; the endpoint $x = 1$ is a null exception. Also

$$0 \le \frac{x^n}{1+x} \le 1,$$

and $g(x) = 1$ lies in $L^1[0,1]$. DCT gives

$$\boxed{\lim_{n\to\infty}\int_0^1 \frac{x^n}{1+x}\,dx = 0}.$$

No antiderivative is needed.

EXAMPLE 6: INTEGRATE A SERIES

For $0 \le x < 1$, $\sum_{k=0}^{n} x^k \uparrow (1-x)^{-1}$. On $[0,a]$ with $a < 1$, MCT yields

$$\begin{aligned}\int_0^a \frac{1}{1-x}\,dx &= \sum_{k=0}^{\infty}\int_0^a x^k\,dx \\ &= \sum_{k=0}^{\infty}\frac{a^{k+1}}{k+1} \\ &= \boxed{-\log(1-a)}.\end{aligned}$$

Nonnegative partial sums make MCT automatic.

A REUSABLE DOMINATED-CONVERGENCE PROOF

Write four sentences: (1) $f_n \to f$ pointwise a.e.; (2) every f_n is measurable; (3) $|f_n| \le g$ a.e.; (4) $g \in L^1$. Therefore DCT gives $\int f_n \to \int f$. If any sentence is missing, the interchange is not yet justified.

Do not dominate by a constant on an infinite-measure space. The function 1 is integrable on $[0,1]$, but not on $\mathbb{R}$. Domination is global and measure-dependent.

When may the order be changed?

Product measure turns area into repeated one-dimensional measurement. Tonelli handles nonnegative functions; Fubini handles absolutely integrable signed functions.

Tonelli: if $f \geq 0$ is measurable, then

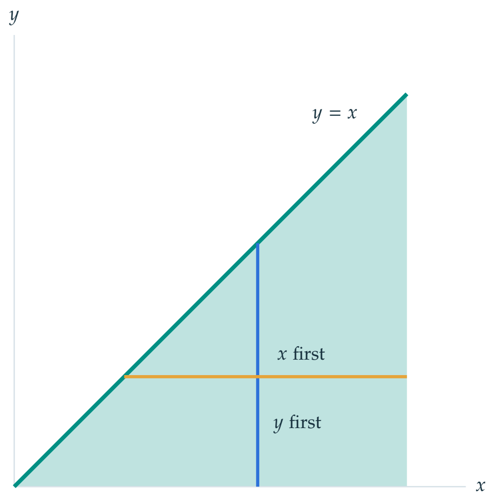
$$\int_{X\times Y} f \, \mathrm{d}(\mu \times \nu) = \int_X \left(\int_Y f(x,y) \, \mathrm{d}\nu(y) \right) \mathrm{d}\mu(x),$$

with values possibly infinite. **Fubini:** if $\int_{X\times Y} |f| < \infty$, both iterated integrals exist a.e., are integrable, and equal the double integral.

EXAMPLE 7: A TRIANGULAR REGION

Let $T = \{(x,y) : 0 < x < 1,\ 0 < y < x\}$. Since $\mathbf{1}_T \geq 0$, Tonelli allows either order:

$$\begin{aligned} m_2(T) &= \int_0^1 \int_0^x 1 \, \mathrm{d}y \, \mathrm{d}x = \int_0^1 x \, \mathrm{d}x = \tfrac{1}{2}, \\ &= \int_0^1 \int_y^1 1 \, \mathrm{d}x \, \mathrm{d}y = \int_0^1 (1-y) \, \mathrm{d}y = \boxed{\tfrac{1}{2}}. \end{aligned}$$



Warning. Conditional cancellation can make two iterated integrals disagree or one fail to exist. For signed f, check $\int |f| < \infty$ before exchanging order.

Integrable functions form a complete space

Lebesgue integration is not only a number-producing device. It creates the normed space L^1, where convergence in mean becomes a robust language for approximation.

$$L^1(X,\mu) = \{f : f \text{ measurable and } \|f\|_1 = \int_X |f| \, d\mu < \infty\}/\text{a.e. equality}.$$

Core properties

- $\|f\|_1 \geq 0$, with equality exactly when $f = 0$ a.e.
- $\|af\|_1 = |a|\|f\|_1$.
- $\|f+g\|_1 \leq \|f\|_1 + \|g\|_1$.
- $|\int f| \leq \int |f| = \|f\|_1$.
- L^1 is complete: every Cauchy sequence has an L^1 limit.

Absolute continuity of the integral
If $f \in L^1$, then for every $\varepsilon > 0$ there is $\delta > 0$ such that

$$\mu(A) < \delta \implies \int_A |f| \, d\mu < \varepsilon.$$

An integrable function cannot hide a fixed amount of mass in sets whose measure tends to zero. This explains why the spike sequence on page 9 has no L^1 dominator.

FINAL CHECKLIST

1. Name the measure space and the set of integration.
2. Establish measurability; split into positive and negative parts if needed.
3. Test $\int |f|$ when a finite signed answer or Fubini is required.
4. Calculate by simple sums, classical calculus, layer cake, or iteration.
5. For limits, state pointwise/a.e. convergence and verify MCT, Fatou, or DCT hypotheses explicitly.
6. Sanity-check sign, scale, null-set invariance, and possible infinite values.

A compact map and practice set

measurable sets $A \in \Sigma$ → indicators $\mathbf{1}_A$ → simple functions $\sum a_k \mathbf{1}_{A_k}$ → $f \geq 0$ $s_n \uparrow f$ → signed f $f^+ - f^-$

One sentence. Measure assigns size to sets; measurable functions pull value thresholds back to measurable sets; simple functions weight those sets; arbitrary nonnegative functions are limits from below; signed integrable functions are differences with finite total absolute mass.

Practice

1. Compute $\int_{[0,2]} (2\mathbf{1}_{[0,1]} + 5\mathbf{1}_{(1,2]})\,\mathrm{d}x$.
2. Find $\int_0^1 \mathbf{1}_{\mathbb{R}\setminus\mathbb{Q}}\,\mathrm{d}x$.
3. For which p is x^{-p} integrable on $(0,1)$? On $(1,\infty)$?
4. Justify $\lim_n \int_0^1 x^n \sin x\,\mathrm{d}x = 0$.
5. Use Tonelli to compute the area under $y = 1 - x^2$ on $[0,1]$.

Answers

1. $2(1) + 5(1) = 7$.
2. 1, because the irrationals have full measure.
3. $p < 1$ near zero; $p > 1$ at infinity.
4. Pointwise a.e. convergence to 0 and domination by $1 \in L^1[0,1]$; apply DCT.
5. $\int_0^1 (1 - x^2)\,\mathrm{d}x = 2/3$.

REFERENCES FOR THE NEXT STEP

G. Folland, *Real Analysis*, 2nd ed.; H. Royden and P. Fitzpatrick, *Real Analysis*, 4th ed.; T. Tao, *An Introduction to Measure Theory*; W. Rudin, *Real and Complex Analysis*, 3rd ed.

This guide emphasizes the conceptual and computational core. Standard results are stated for complete measure spaces where convenient; consult the references for construction of Lebesgue measure, completion, and full proofs.