

Combinatorics

Permutations, Combinations & Binomial Identities

A reference sheet • conventions: $n, k, m \in \mathbb{Z}$, $\binom{n}{k} = 0$ when $k < 0$ or $k > n \geq 0$

1 Foundations

Rules of counting

Product. A task in r independent stages with $n_1, \dots, n_r$ choices: $n_1 n_2 \cdots n_r$ outcomes.

Sum. Disjoint cases $A_1, \dots, A_r$: $|\cup A_i| = \sum |A_i|$.

Bijection. If $f: A \rightarrow B$ is a bijection then $|A| = |B|$. *The workhorse: recount the hard set as an easy one.*

Double counting. Count one set two ways $\Rightarrow$ an identity.

Pigeonhole. n objects in m boxes $\Rightarrow$ some box holds $\geq \lceil n/m \rceil$.

Factorial

$$n! = \prod_{i=1}^n i, \quad 0! = 1, \quad n! = n \cdot (n-1)!$$

$$\Gamma(n+1) = n!, \quad \Gamma(z+1) = z\Gamma(z), \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

2 Permutations and combinations

Ordered selections (k -permutations)

$$P(n, k) = {}^n P_k = \frac{n!}{(n-k)!} = n^{\overline{k}}$$

$$n^{\overline{k}} = \underbrace{n(n-1)(n-2) \cdots (n-k+1)}_{k \text{ factors}}$$

Falling factorial. $n^{\overline{0}} = 1$, $n^{\overline{n}} = n!$.

Unordered selections

$$\binom{n}{k} = {}^n C_k = \frac{n!}{k!(n-k)!} = \frac{P(n, k)}{k!} = \frac{n^{\overline{k}}}{k!}$$

$$P(n, k) = k! \binom{n}{k} \quad \text{order matters} \Rightarrow \text{multiply by } k!$$

Rising factorial (for series work)

$$x^{\overline{k}} = x(x+1) \cdots (x+k-1), \quad x^{\underline{k}} = (-1)^k (-x)^{\overline{k}}$$

Generalised binomial coefficient

For any $x \in \mathbb{C}$ and $k \in \mathbb{Z}_{\geq 0}$:

$$\binom{x}{k} := \frac{x^{\underline{k}}}{k!} = \frac{x(x-1) \cdots (x-k+1)}{k!}$$

This is the definition that makes Newton's series and negative-index identities work.

3 The four sampling problems

Choose k items from n distinct types:

	no repeats	repeats ok
ordered	$\frac{n!}{(n-k)!}$	n^k
unordered	$\binom{n}{k}$	$\binom{n+k-1}{k}$

Bottom-right is the multiset coefficient $\binom{n}{k} = \binom{n+k-1}{k} = \binom{n+k-1}{n-1}$.

Permutations of a multiset

n objects with $n_1, \dots, n_r$ copies of each type, $\sum n_i = n$:

$$\binom{n}{n_1, n_2, \dots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!}$$

Circular arrangements

n distinct objects round a circle: $(n-1)!$

if reflections identified (necklace): $\frac{(n-1)!}{2}$ for $n \geq 3$.

4 Core identities

$$\binom{n}{k} = \binom{n}{n-k} \quad \text{symmetry}$$

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad \text{Pascal}$$

$$k \binom{n}{k} = n \binom{n-1}{k-1} \quad \text{absorption}$$

$$(n-k) \binom{n}{k} = n \binom{n-1}{k} \quad \text{co-absorption}$$

$$\binom{n}{k} = \frac{n-k+1}{k} \binom{n}{k-1} \quad \text{step ratio}$$

$$\binom{n}{k} \binom{k}{j} = \binom{n}{j} \binom{n-j}{k-j} \quad \text{subset-of-subset}$$

$$\binom{-n}{k} = (-1)^k \binom{n+k-1}{k} \quad \text{upper negation}$$

Subset-of-subset ("trinomial revision") is the single most useful rewriting rule: it lets you pull a factor out of a double sum.

5 Binomial theorem & friends

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}, \quad n \in \mathbb{Z}_{\geq 0}$$

$$(1 + x)^n = \sum_{k=0}^n \binom{n}{k} x^k$$

Newton's generalised binomial series

For $\alpha \in \mathbb{C}$ and $|x| < 1$:

$$(1 + x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

$$\frac{1}{(1-x)^m} = \sum_{k=0}^{\infty} \binom{m+k-1}{k} x^k \quad m \in \mathbb{Z}_{\geq 1}$$

$$\sqrt{1-4x} = 1 - 2 \sum_{k \geq 1} \frac{1}{k} \binom{2k-2}{k-1} x^k$$

Multinomial theorem

$$(x_1 + \dots + x_m)^n = \sum_{\substack{k_1 + \dots + k_m = n \\ k_i \geq 0}} \frac{n!}{k_1! \dots k_m!} x_1^{k_1} \dots x_m^{k_m}$$

Number of distinct terms: $\binom{n+m-1}{m-1}$. Sum of all coefficients: m^n .

7 Summation identities — rows

Row sums

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = [n=0]$$

$$\sum_{k \text{ even}} \binom{n}{k} = \sum_{k \text{ odd}} \binom{n}{k} = 2^{n-1} \quad (n \geq 1)$$

Weighted row sums

$$\sum_{k=0}^n k \binom{n}{k} = n 2^{n-1}$$

$$\sum_{k=0}^n k^2 \binom{n}{k} = n(n+1) 2^{n-2}$$

$$\sum_{k=0}^n \frac{1}{k+1} \binom{n}{k} = \frac{2^{n+1} - 1}{n+1}$$

$$\sum_{k=0}^n \frac{(-1)^k}{k+1} \binom{n}{k} = \frac{1}{n+1}$$

$$\sum_{k=0}^n (-1)^k k^m \binom{n}{k} = \begin{cases} 0, & 0 \leq m < n \\ (-1)^n n!, & m = n \end{cases}$$

6 Worked micro-examples

- **Handshakes.** Pairs from n people: $\binom{n}{2} = \frac{n(n-1)}{2}$.
- **Subsets.** A set of size n has 2^n subsets, $\binom{n}{k}$ of size k .
- **Lattice paths.** Monotone paths $(0,0) \rightarrow (m,n)$ on the grid: choose which of the $m+n$ steps go right, $\binom{m+n}{m}$.
- **Binary strings.** Length n with exactly k ones: $\binom{n}{k}$.
Same object as the line above — that is the bijection principle at work.
- **Five-card hands.** $\binom{52}{5} = 2\,598\,960$. Full houses: $13 \binom{4}{3} \cdot 12 \binom{4}{2} = 3\,744$. *Order the choices so nothing is double counted: rank of the triple first.*
- **Anagrams of MISSISSIPPI.** Letters $M^1 I^4 S^4 P^2$: $\frac{11!}{1! 4! 4! 2!} = 34\,650$.
- **Identical balls.** 10 identical balls into 4 distinct boxes: $\binom{10+3}{3} = 286$; with every box non-empty, $\binom{9}{3} = 84$.
- **Committee with a chair.** k people from n , one of them chair: $n \binom{n-1}{k-1} = k \binom{n}{k}$ — *absorption, proved by double counting.*

7 Summation identities — columns & convolutions

Column sums — “hockey stick”

$$\sum_{i=k}^n \binom{i}{k} = \binom{n+1}{k+1}, \quad \sum_{i=0}^m \binom{n+i}{i} = \binom{n+m+1}{m}$$

Vandermonde convolution

$$\sum_k \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$$

Special cases ($r = m = n$, using symmetry):

$$\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}, \quad \sum_{k=0}^n k \binom{n}{k}^2 = n \binom{2n-1}{n-1}$$

Shallow diagonals

$$\sum_{k \geq 0} \binom{n-k}{k} = F_{n+1} \quad \text{Fibonacci, } F_1 = F_2 = 1$$

8 Compositions & stars and bars

Non-negative integer solutions of $x_1 + \dots + x_k = n$:

$$\binom{n+k-1}{k-1} = \binom{n+k-1}{n}$$

Strictly positive solutions ($x_i \geq 1$):

$$\binom{n-1}{k-1}$$

With floors $x_i \geq a_i$: substitute $y_i = x_i - a_i$,

$$\binom{n - \sum a_i + k - 1}{k - 1}$$

Inequality version $x_1 + \dots + x_k \leq n$: add a slack variable $\Rightarrow \binom{n+k}{k}$.

Ceilings ($x_i \leq b_i$) are not a closed form — handle them with inclusion–exclusion (§9) over which variables overflow.

9 Inclusion–exclusion

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{\emptyset \neq S \subseteq [n]} (-1)^{|S|+1} \left| \bigcap_{i \in S} A_i \right|$$

$$\left| \overline{A_1} \cap \dots \cap \overline{A_n} \right| = \sum_{S \subseteq [n]} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right|$$

Derangements (D_n : no fixed point)

$$D_n = n! \sum_{k=0}^n \frac{(-1)^k}{k!} = \left\lfloor \frac{n!}{e} + \frac{1}{2} \right\rfloor \quad (n \geq 1)$$

$$D_n = n D_{n-1} + (-1)^n = (n-1)(D_{n-1} + D_{n-2})$$

$$D_0 = 1, D_1 = 0, D_2 = 1, D_3 = 2, D_4 = 9, D_5 = 44$$

Permutations of $[n]$ with exactly r fixed points:

$$\binom{n}{r} D_{n-r}.$$

Surjections $[n] \twoheadrightarrow [m]$

$$\text{Surj}(n, m) = \sum_{j=0}^m (-1)^j \binom{m}{j} (m-j)^n = m! \left\{ \begin{matrix} n \\ m \end{matrix} \right\}$$

Burnside / orbit counting

Number of orbits under a group G acting on X :

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|$$

Necklaces, colourings up to rotation/reflection.

10 The twelvefold way — placing n balls into k boxes

balls	boxes	any	at most one per box	at least one per box
distinct	distinct	k^n	$k^{\underline{n}} = \frac{k!}{(k-n)!}$	$k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\}$
identical	distinct	$\binom{n+k-1}{n}$	$\binom{k}{n}$	$\binom{n-1}{k-1}$
distinct	identical	$\sum_{j=1}^k \left\{ \begin{matrix} n \\ j \end{matrix} \right\}$	$[n \leq k]$	$\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$
identical	identical	$p_{\leq k}(n)$	$[n \leq k]$	$p_k(n)$

$p_k(n)$ = partitions of n into exactly k positive parts; $p_{\leq k}(n)$ = into at most k parts. Recurrence: $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$.

11 Stirling numbers

Stirling numbers, second kind $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$

Partitions of $[n]$ into k non-empty blocks.

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = k \left\{ \begin{matrix} n-1 \\ k \end{matrix} \right\} + \left\{ \begin{matrix} n-1 \\ k-1 \end{matrix} \right\}, \quad \left\{ \begin{matrix} 0 \\ 0 \end{matrix} \right\} = 1$$

$$\left\{ \begin{matrix} n \\ k \end{matrix} \right\} = \frac{1}{k!} \sum_{j=0}^k (-1)^j \binom{k}{j} (k-j)^n \quad x^n = \sum_k \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^{\underline{k}}$$

11 Stirling (first kind), Bell & Catalan

Stirling numbers, first kind (unsigned) $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$

Permutations of $[n]$ with exactly k cycles.

$$\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] = (n-1) \left[\begin{smallmatrix} n-1 \\ k \end{smallmatrix} \right] + \left[\begin{smallmatrix} n-1 \\ k-1 \end{smallmatrix} \right] \quad x^{\overline{n}} = \sum_k \left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right] x^k$$

Bell numbers

$$B_n = \sum_{k=0}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}, \quad B_{n+1} = \sum_{k=0}^n \binom{n}{k} B_k$$

$$\sum_{n \geq 0} \frac{B_n}{n!} x^n = e^{e^x - 1}$$

1, 1, 2, 5, 15, 52, 203, 877, 4140, ...

Catalan numbers

$$C_n = \frac{1}{n+1} \binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}$$

$$C_n = \frac{2(2n-1)}{n+1} C_{n-1}, \quad C_0 = 1$$

$$C_{n+1} = \sum_{i=0}^n C_i C_{n-i}, \quad \sum_{n \geq 0} C_n x^n = \frac{1 - \sqrt{1-4x}}{2x}$$

1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, ...

Counts: balanced bracketings of n pairs; monotone lattice paths from $(0,0)$ to (n,n) staying weakly below the diagonal; triangulations of an $(n+2)$ -gon; binary trees on n nodes.

12 Generating-function dictionary

$$\binom{n}{k} = [x^k] (1+x)^n \quad \binom{n+k}{k} = [x^k] \frac{1}{(1-x)^{n+1}}$$

sequence a_n	$\sum a_n x^n$
1	$\frac{1}{1-x}$
$\binom{m}{n}$	$(1+x)^m$
$\binom{n+m-1}{n}$	$\frac{1}{(1-x)^{m+1}}$
C_n (Catalan)	$\frac{1 - \sqrt{1-4x}}{2x}$
$\binom{2n}{n}$	$\frac{1}{\sqrt{1-4x}}$
F_n (Fibonacci)	$\frac{x}{1-x-x^2}$
$D_n/n!$ (EGF)	$\frac{e^{-x}}{1-x}$
$\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ (EGF in n)	$\frac{(e^x - 1)^k}{k!}$

Products of GFs = convolutions = “choose how to split the resource”. This is how Vandermonde and the Catalan recurrence both fall out in one line.

13 Arithmetic of binomial coefficients

Legendre's formula

The exponent of the prime p in $n!$ is

$$v_p(n!) = \sum_{i \geq 1} \left\lfloor \frac{n}{p^i} \right\rfloor = \frac{n - s_p(n)}{p-1}$$

$s_p(n)$ = digit sum of n in base p .

Kummer's theorem

$v_p \left(\binom{m+n}{m} \right)$ = number of carries when adding m and n in base p .

Lucas' theorem

With base- p digits $m = \sum m_i p^i$, $r = \sum r_i p^i$:

$$\binom{m}{r} \equiv \prod_i \binom{m_i}{r_i} \pmod{p}$$

Corollary: $\binom{m}{r}$ is odd iff r 's binary digits are a submask of m 's, i.e. $r \& m = r$.

Useful divisibilities

$$p \mid \binom{p}{k} \quad (0 < k < p), \quad n+1 \mid \binom{2n}{n}$$

$$\binom{2n}{n} = 2 \binom{2n-1}{n-1}, \quad \frac{n^k}{k!} \in \mathbb{Z} \text{ always}$$

14 Estimates and asymptotics

Stirling's approximation

$$n! = \sqrt{2\pi n} \left(\frac{n}{e} \right)^n \left(1 + \frac{1}{12n} + \frac{1}{288n^2} + O(n^{-3}) \right)$$

$$\ln n! = n \ln n - n + \frac{1}{2} \ln(2\pi n) + O(n^{-1})$$

Bounds on $\binom{n}{k}$

$$\left(\frac{n}{k} \right)^k \leq \binom{n}{k} \leq \frac{n^k}{k!} \leq \left(\frac{en}{k} \right)^k$$

$$\sum_{i=0}^k \binom{n}{i} \leq 2^{nH(k/n)} \quad (0 < k \leq n/2)$$

with binary entropy $H(q) = -q \log_2 q - (1-q) \log_2 (1-q)$.

Central binomial

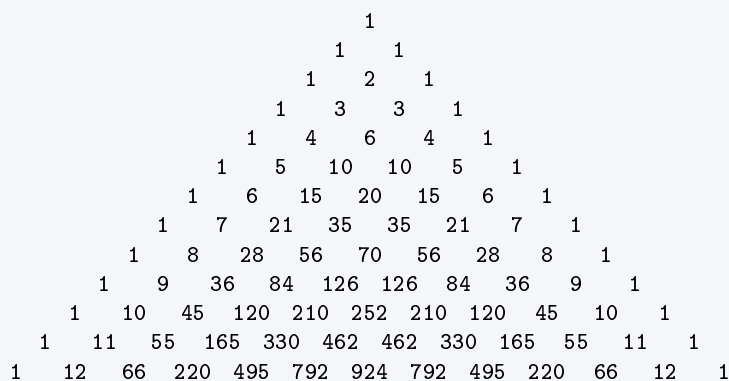
$$\binom{2n}{n} \sim \frac{4^n}{\sqrt{\pi n}}, \quad \frac{4^n}{2\sqrt{n}} \leq \binom{2n}{n} \leq \frac{4^n}{\sqrt{\pi n}}$$

$$C_n \sim \frac{4^n}{n^{3/2} \sqrt{\pi}}$$

Fixed k , large n

$$\binom{n}{k} = \frac{n^k}{k!} \left(1 + O\left(\frac{k^2}{n} \right) \right)$$

15 Pascal's triangle, rows 0–12



Row n lists $\binom{n}{0}, \dots, \binom{n}{n}$: it sums to 2^n , alternates to 0, and each entry is the sum of the two above it. A diagonal summed downwards lands on the entry below-right of where it stopped (hockey stick); shallow diagonals sum to Fibonacci numbers; the middle of an even row is the central binomial $\binom{2m}{m}$.

16 Numerical reference

n	$n!$	2^n	$\binom{2n}{n}$	C_n	B_n	D_n
0	1	1	1	1	1	1
1	1	2	2	1	1	0
2	2	4	6	2	2	1
3	6	8	20	5	5	2
4	24	16	70	14	15	9
5	120	32	252	42	52	44
6	720	64	924	132	203	265
7	5 040	128	3 432	429	877	1 854
8	40 320	256	12 870	1 430	4 140	14 833
9	362 880	512	48 620	4 862	21 147	133 496
10	3 628 800	1 024	184 756	16 796	115 975	1 334 961

C_n Catalan, B_n Bell, D_n derangements.

17 Choosing the right tool

Ask three questions

- Does order matter?**
yes $\Rightarrow P(n, k)$ or n^k ; no $\Rightarrow \binom{n}{k}$ or $\left(\!\!\binom{n}{k}\!\!\right)$.
- Are repeats allowed?**
decides which column of §3 you are in.
- Are the objects distinguishable?**
identical objects $\Rightarrow$ stars and bars or partitions, not powers.

Standard moves

- **Complement.** Count the bad cases and subtract; “at least one” almost always means $1 - \Pr(\text{none})$.
- **Glue.** Objects forced adjacent: treat the block as one item, then multiply by the internal arrangements.
- **Gap method.** Objects forced non-adjacent: arrange the others ($m!$), then choose gaps, $\binom{m+1}{k}k!$.
- **Overcount, then divide.** Count with labels, divide by the size of the symmetry group.
- **Bijection.** Recode the objects (binary strings, lattice paths, bracket sequences).
- **Recurse.** Condition on the last element; Pascal, Catalan and Stirling recurrences all arise this way.

Frequent traps

- Dividing by $k!$ when the k items were never ordered.
- Forgetting that identical objects kill the $n!$.
- Double counting in “at least” problems — use inclusion–exclusion, not a single subtraction.
- Circular problems: fix one object to break the rotational symmetry.
- Using $\binom{n}{k}$ with $k > n$: it is 0, not an error.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad \text{— when in doubt, condition on whether the last element is in.}$$